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Herbert Shaw, Royal College of Science, South Kensington, S.W. (proposed by A. Fowler); and

James Henry Worthington, Student in the University of Oxford, Bindon, Wellington, Somerset (proposed by H. H. Turner).

Sixty presents were announced as having been received since the last meeting, including, amongst others:—

E. T. Whittaker, The Theory of Optical Instruments, presented by the author; framed photograph of 14th century clock restored by J. J. Hall, presented by Mr. Hall; Calendrier Perpétuel, presented by M. Bosson.

Seven transparencies of Sun-spots, Comet, Planets, and Nebulæ, from negatives taken at the Royal Observatory, Greenwich, presented by the Astronomer Royal.

J. C. Kapteyn, On the number of stars of determined magnitude and determined galactic latitude (Pub. Astron. Laboratory, Groningen, No. 18), presented by Professor Kapteyn.

On the Lunar Inequalities due to Planetary Action.

By Ernest W. Brown, D.Sc., F.R.S.

1. I have lately finished the computation of the terms in the Moon's motion due to the actions—direct and indirect—of the planets, and it is of interest to compare the results with those of previous investigators. There are two extensive lists in print: one given by Radau[*] in 1892, and the other by Newcomb[†] a few months ago. Both of these give only the terms in longitude, and I shall therefore limit the results set forth here for discussion to these terms, reserving the complete list for a memoir, containing also my methods and the details, which I hope to present to the Society at a later date. I have also omitted here the secular terms and those which depend solely on the lunar arguments, since the results for them are not in doubt, and since they will ultimately be combined with terms arising from the figure of the Earth. Further, I only consider quantities of the first order relative to the planetary masses, while Newcomb has included the mutual perturbations in his values for the direct action. But the latter do not affect any of the terms except one or two of long period, mentioned in No. 10 below.

The great majority (about 400) of the terms are of short period,

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[*] "Recherches concernant les Inégalités planétaires du mouvement de la Lune," *Paris Obs. Ann.* (*Mém.*), vol. xxi.

[†] "Investigation of Inequalities in the motion of the Moon produced by the action of the planets," *Carnegie Inst. Publ.*, 72.

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and are most conveniently added to the true longitude. The terms of very long period are best added to the mean longitude, and they are so set down here; but the changes in the other elements for these long period terms are not given, since they can be exhibited as simple ratios to the mean longitude portions. Radau adds these long period terms to the true longitude, and I have therefore dropped the corresponding short period terms which he gives.

In his final results, Newcomb drops coefficients less than 0″·003. I have done the same in transcribing from my manuscripts, except occasionally where a smaller coefficient comes in the middle of a long series, or where Newcomb or Radau give a coefficient greater than 0″·002. All my coefficients have been computed to 0″·001 except those of a few long period terms in δw_1; these exceptions are noted by a comma placed before the zero, which is set down as the last figure in order not to break the continuity of the representation.

2. The notation for the arguments of the tables is as follows:—

l, 2D, F, the mean anomaly, the argument of the "variation" and the principal argument of the latitude of the Moon.
V*, the Moon's true longitude.
w_1, the Moon's mean longitude.
Q, V, T, M, J, S, the mean longitudes of Mercury, Venus, the Earth, Mars, Jupiter, and Saturn.
$h = \mathrm{D} - \mathrm{F} + \mathrm{T}$, the mean longitude of the Moon's node.

The results are arranged in six columns. The first pair, headed B, contains the coefficient and angle as computed by myself; the second pair, headed N, those of Newcomb; the third pair, headed R, those of Radau. The coefficients are given in units of 0″·001. The angle has, except in δw_1, been divided into four parts for convenience: ϕ contains the lunar arguments l, D, F, h; a, the constant part of the angle so taken that the coefficients are all positive; j, j' the multiples of T − P and T (or j, j'' the multiples of P − T and P) so that j' (or j'') represents the order of the term with reference to the eccentricities and inclination of the Earth and planets. The values of j' (or j'') and ϕ, once given, are understood to be the same for every following term until new values are set down. The differences between the results of Newcomb and myself are marked with the letters E_1, E_2, E_3, U whenever they exceed 0″·020, and they are considered in No. 8 below.

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* An italic capital to distinguish from the mean longitude of Venus.

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$$\delta V = +0''\cdot001\ \mathrm{C} \sin\{\phi + j(\mathrm{T}-\mathrm{V}) + j'\mathrm{T} + \alpha\},\ \text{Venus.}$$

$\phi=0$		B		N		R		
j	j'	C	α	C	α	C	α	
1	0	822	0°·0	882	0°·1	860	0°	E_1
2		307	179·8	401	179·3	283	180	E_1
3		42	359·3	37	1·6	65	0	
4		46	0					
5		33	0					
6		24	0					
7		17	0					
8		12	0					
9		8	0					
10		6	0					
11		4	0					
21		5	0					
−2	1	10	254					
−1		16	84	14	85			
1		42	82	55	81·5			
2		348	272·9	354	276·2	348	275	
3		176	271·7	197	272·2	181	272	E_1
4		2	82					
5		4	271					
6		6	272					
7		4	272					
−1	2	3	27					
1		5	25					
2		3	33	5	34			
3		92	199·0	82	208·0	80	198	
4		26	204					
5		9	17					
6		4	207					
5	3	26	114	23	112			

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2D		B		N		R		
j	j′	C	α	C	α	C	α	
			°		°		°	
−11	0	3	0					
−10		5	0					
− 9		6	0					
− 8		8	0					
− 7		8	0					
− 6		11	0					
− 5		11	0					
− 4		8	0					
− 3		34	180·0	56	359	32	180	E_1
− 2		36	0·0	24	0·0	48	0	
− 1		23	180	8	180	30	180	
1		99	0·0	96	0·0	105	0	
2		136	179·5	143	178·8	148	180	
3		13	178	35	180	23	180	E_1
4		4	180					
18		3	0					
1	1	3	232					
2		40	271·0	102	271·8	46	270	E_2
3		37	271·5	92	272·4	44	270	E_2
4		5	89					
20		10	273					
−11	−1	3	78					
−10		4	78					
− 9		4	78					
− 8		4	78					
− 7		5	78					
− 6		5	78					
− 5		11	82					
− 4		7	78					
− 3		2	288	5	268·2	12	270	
− 2		19	271	49	267·8	28	270	E_2
1		9	98					
2		13	281					
3		3	281					

2D		B		N		R		
j	j'	c	α	c	α	c	α	
			°		°		°	
3	2	11	199	26	199·7			
4		3	202					
5		4	20					
− 6	− 2	6	162					
− 3		15	342	35	339·8			
− 2		3	7					
15		4	151					
l								
− 7	0	4	180					
− 6		5	180					
− 5		6	180					
− 4		6	180					
− 3		8	180	6	180	10	180	
− 2		61	0·0	71	0·6			
− 1		129	180·0	149	180·0	66	180	E_1
1		152	0·0	166	0·0	80	0	
2		48	180·0	93	180			E_1
3		127	180·0	8	0	138	180	E_1
4		11	180					
21		30	0			56	0	
− 2	1	4	258					
− 1		1	79	4	80			
1		8	75	3	80			
2		46	271	82	273·1			E_1
3		40	272	41	271·8			
4		5	272					
5		4	92					
23		6	272			12	273	
− 3	− 1	32	268	37	267·0			
− 2		46	264	80	266·5			E_1
− 1		9	104	3	100			
1		3	102	4	280			
2		7	282					
3		7	280					

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l		B		N		R	
j	*j′*	C	*a*	C	*a*	C	*a*
			°		°		°
3	2	14	210	23	201		
4		4	205				
5		3	19				
6		16	198				
−5	−2	3	161				
−4		4	336	22	336		
−3		15	331				
5	3	4	115	5	115		
−5	−3	4	65	5	65		
2D−*l*							
−10	0	3	180				
−9		6	180				
−8		8	180				
−7		13	180				
−6		22	180				
−5		38	180				
−4		83	180·0				
−3		658	180·0	646	179·8	681	180
−2		137	0·0	142	0·0	192	0
−1		13	180	25	180	42	180
1		133	0·0	146	0·0	143	0
2		157	179·6	174	180·0	172	180
3		14	178	14	180	18	180
4		3	180				
18		11	0			12	0
−2	1	3	259				
−1		3	270				
2		65	271·4	71	269·5	70	270
3		49	271·9	53	273·7	50	270
4		5	90				
20		126	273·0			140	272

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2D − l		B		N		R	
j	j'	C	a	C	a	C	a
			°		°		°
− 8	− 1	3	269				
− 7		5	269				
− 6		8	269				
− 5		24	269	25	268		
− 4		30	89				
− 3		0	...			13	270
− 2		38	268	41	264·0	45	270
1		9	101				
2		13	281				
17		3	253				
3	2	19	200	21	202	22	198
4		4	202				
5		4	19				
− 7	− 2	3	340				
− 6		79	162·6	83	161·3	82	163
− 5		4	165				
− 3		24	340	26	337	27	342
− 2		3	6				
15		25	151				
5	3	4	114				
12	− 4	33	303				
2D + l							
− 5		3	0				
− 3		4	0				
− 2		4	0				
1		11	0	6	180		
2		15	180	9	0		
3		3	180				
2	1	5	271				
3		4	271				
− 2	− 1	3	269				

		B		N		R	
$2l$ j	j'	C	a	C	a	C	a
			°		°		°
− 2	0	5	0				
− 1		10	180				
1		10	0				
2		6	180				
3		9	180				
2	1	3	273				
− 2	− 1	3	267				
$2l-2D$							
− 2	0	7	0	9	0		
− 1		5	180	7	180		
2		3	180	8	180		
3		73	180·0	72	180·0	75	180
4		3	0				
4	1	4	92				
− 20	− 1	7	267				
− 3		3	268				
− 2		3	268	3	261		
6	2	62	17·4	62	17·5	63	17
$4D-l$							
1	0	6	180				
2		7	0				
$3l-2D$							
3	0	3	180	3	180		
6	2	5	18	3	26		
$2l-4D$							
3	0	8	0				
$-4D$							
3	0	7	0				
$-D$							
3	1	5	273			31	91·4
$l-D$							
3	1	11	273			48	91·4
$2l+2D$							
2	0	0		3	0		

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h		B		N		R	
j	*j′*	C	*α*	C	*α*	C	*α*
−5	−2	3	255°				
−4		9	255				
−3		16	75			16	256
−2		5	75				
h+2F							
−2	−3	3	75				

$$\delta w_1 = +0''\cdot001 \text{ C} \sin (\phi + \alpha).$$

	B		N		R		
ϕ	C	*α*	C	*α*	C	*α*	
$13T-8V$	237	313°·9	247	316°·1	250	318°·7	
$l+16T-18V$	1449,0	151·0	1477,0	150·1	1442,0	150	E_3
$l+3T-10V$	35,0	33					
$l+29T-26V$	108	112·0			110	121	
$2F-2D+6T-5V$	54	270·0			60	269	
$3l-2D+24(T-V)$	10	0			12	0	
$D+l-F+17T-18V$	8	75					
$F+24T-23V$	3	285			8	0	
$D+12T-15V$	13	262			16	261	
$D+25T-23V$	13	190			19	194·5	
$3D-2F+18T-18V$	2	272			8	0	
$2D-2F-l-22T+23V$	< 10				4	284	
$D-2l-20T+21V$	0				3	288	

$$\delta w_1 = +0''\cdot001 \text{ C} \sin (\phi + \alpha), \text{ Mercury.}$$

$Q-4T$	3	239°				
$2D-l-3Q+T$	75	105			57	150°
$2D-l-4Q+5T$	3	113			6	105
$l-2F+4Q-3T$	3	113			7	105

$$\delta V = +0''.001\ C \sin\{\phi + j(M - T) + j''M + \alpha\},\ \text{Mars.}$$

$\phi=0$		B		N		R		
j	j''	C	α	C	α	C	α	
1	0	11	180°	11	174°	14	180°	
2		195	180·2	224	180	228	180	E_1
3		14	357	14	0	16	0	
4		5	349					
−3	1	6	260					
0		0		30	82			U
1		327	224·4	372	229·2	422	229	E_1
2		38	212·4	42	213·6	53	213	
3		48	212·5	47	212·9	55	212	
4		10	331					
2	2	93	244·8	95	249·8	112	247	
3		20	245	30	120			U
4		14	244					
5		6	62					
3	3	16	277					
4		13	276					
5		6	275					
6		3	94					
2D								
−2	0	5	0	6	0	16	0	
1		4	180					
2		44	181	50	180·0	59	180	
3		5	0					
1	1	23	224	52	225·0	30	224	E_2
2		6	212					
3		8	214	28	214			E_2
4		3	37					

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2D		B		N		R		
j	*j''*	c	*a*	c	*a*	c	*a*	
			°		°		°	
– 5	– 1	3	149					
– 3		1	327	5	333			
– 2		3	328					
– 1		23	317	58	316·8	34	315	E_2
3		3	280					
2	2	5	244					
3		4	244					
4		4	246					
– 2	– 2	8	297					
l								
– 3	0	3	180	3	180			
– 2		38	0	43	0·0			
– 1		4	0	4	0			
1		5	180	4	180			
2		43	180	48	180			
3		3	0	3	0			
4		3	180					
0	1	0		3	80			
1		73	223·3	55	219·9	35	230	
2		10	212	9	215			
3		13	213	10	215			
5		8	198					
– 3	– 1	9	330	10	324			
– 2		8	327	9	324			
– 1		74	306·3	55	318·9	36	310	
0		0		3	99			
2	2	17	245	17	244			
3		5	245					
4		3	244					
6		6	63					
– 4	– 2	3	296					
– 3		5	295					
– 2		18	295	17	296			
26		0				12	306	

2D − *l*		B		N		R	
j	*j'*	C	*a*	C	*a*	C	*a*
			°		°		°
− 5	0	3	180				
− 4		20	182			19	180
− 3		5	0	6	0		
− 2		13	0	17	0	34	0
− 1		3	0	4	0	7	0
1		8	180	5	180	9	180
2		61	181	66	180·0	73	180
3		5	353	3	0		
1	1	31	220	41	226·4	47	223
2		11	212	9	215		
3		14	214	13	215		
4		3	27				
− 6	− 1	3	149				
− 5		43	162			42	148
− 4		3	329				
− 3		3	327	5	333		
− 2		6	328	6	324		
− 1		35	320	43	316·9	51	317
3		4	280				
2	2	11	244	9	244		
3		6	244				
4		5	245				
− 6	− 2	33	298				
− 3		3	296				
− 2		14	297	11	294		
2D + *l*							
2	0	6	180	4	0		
1	1	3	82				
− 1	− 1	3	93				
2*l*							
− 2	0	3	0				
2		3	180				
1	1	3	232				
− 1	− 1	3	308				

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$2l-2D$		B		N		R	
j	j''	C	α	C	α	C	α
−2	0	4	0°	4	0°		°
5	1	17	209			17	212
6	2	18	244				

$$\delta w_1 = +0''\!\cdot\!001 \text{ C} \sin(\phi + \alpha), \text{ Mars.}$$

	B		N		R	
ϕ	C	α	C	α	C	α
$8M-4T$	3	310°				°
$9M-5T$	8	305				
$11M-6T$	6	335				
$13M-7T$	6	19				
$15M-8T$	26	43				
$17M-9T$	4	63				
$4D-3l+25M-23T$	4,0	67				
$h+2M-T$	17	165			19	164
$D-l-4M+3T$	<10				3	153

$$\delta V = +0''\!\cdot\!001 \text{ C} \sin\{\phi + j(\text{J}-\text{T}) + j''\text{J} + \alpha\}, \text{ Jupiter.}$$

$\phi=0$		B		N		R		
j	j''	C	α	C	α	C	α	
1	0	643	178°·8	741	178°·8	646	180°	E_1
2		187	359·6	242	359·8	196	0	E_1
3		10	7	9	6	12	0	
−3	1	6	257	6	80			
−2		18	274	16	275			
0		87	289·9	183	283·0	139	279	U
1		165	241·5	193	242·7	172	242	E_1
2		52	352·0	42	77·7	49	352	U
3		4	355					
−1	2	10	250					
0		5	324					
1		25	238	24	83			U
2		6	344					
1	3	3	230					

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2D		B		N		R		
j	j''	C	a	C	a	C	a	
			°		°		°	
−3	0	4	0	18	180			E_2
−2		70	180·0	18	0	54	180	E_1
−1		33	1·1	16	0	45	0	
1		167	178·5	168	177·9	175	180	
2		85	359·2	92	359·3	88	0	
3		7	13	74	2·3			U
0	1	27	349	36	350·5	32	350	
1		35	236·5	89	237·4	39	238	E_2
2		15	352	42	354·5			E_2
−2	−1	30	8	7	9			E_1
−1		6	313	15	311			
0		33	184	40	181·0	38	189	
2		9	273	38	272·0			E_2
3		6	102	19	105			
1	2	5	236	10	240			
2		3	345					
0	−2	3	200					
1		6	110					
l								
−3	0	1	170	4	180			
−2		36	180	45	180	12	180	
−1		144	1·0	140	1·6	36	0	
1		158	179·0	163	178·6	48	180	
2		190	180·0	64	0·0	165	180	E_1
3		5	21	3	0·0			
−2	1	6	274	6	270			
0		62	282·3	38	7·5			E_1
1		39	242	36	268·4			
2		96	352·5	10	351	65	353	E_1

l		B		N		R		
j	j''	c	a	c	a	c	a	
−2	−1	7	188°	10	189°			
−1		35	298	32	278·6			
0		63	257·2	38	172·5			U
2		6	273	6	270			
3		8	286					
0	2	7	326					
1		5	238	3	206			
2		4	343					
−1	−2	5	302	3	334			
0		7	214					
$2D-l$								
−4	0	4	180					
−3		22	182	18	0			U
−2		1137	180·3	1140	180·2	881	180	
−1		51	0·9	64	0·0	101	0	
1		211	178·4	230	179·0	220	180	
2		89	359·2	98	0·7	89	0	
3		6	14	30	1·9			U
−3	1	5	261					
−2		13	310	18	294			
0		56	5·5	60	353·4	60	351	
1		46	237·0	48	238·7	49	236	
2		20	352	21	353			
−3	−1	6	187					
−2		436	7·5	445	7·5	316	7	
−1		18	296	19	311	25	304	
0		60	174·2	62	182·2	63	184	
2		16	273	18	276			
3		7	102					
1	2	6	237	4	237			
2		3	344					
−2	−2	5	19			9	16	
1		3	291					

$2D+l$		B		N		R		
j	j''	C	α	C	α	C	α	
			°		°		°	
−2	0	3	180					
−1		5	1					
1		21	178	11	0			U
2		7	359	6	180			
0	1	4	353					
1		5	237					
2		3	352					
0	−1	4	182					
2		3	273					
$2l$								
−2	0	3	180					
−1		11	2					
1		12	178					
2		10	180					
0	1	5	293					
1		3	239					
−1	−1	3	301					
0		5	247					
$2l-2D$								
−2	0	5	180	4	180			
−1		11	2	11	0			
1		3	0					
2		240	179·9	256	180·0	206	180	
0	1	0		4	350			
2	1	284	172·5	258	172·3	194	173	U
−1	−1	2	302	3	298			
0	−1	2	186	4	189			
2	2	3	163			2	164	
$4D-l$								
−2	0	7	180					
1		9	358					
2		5	179					
−2	−1	3	7					
$3l-2D$								
2	0	7	180	10	180			
2	1	5	172	15	257			

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		B		N		R	
$2l-4D$							
j	j''	C	α	C	α	C	α
			°		°		°
2	0	9	0				
2	1	7	173				
h							
0	1	4	81			4	75
0	−2	0				11	277

$$\delta V = +0''\cdot001\ \mathrm{C}\sin\{\phi + j(\mathrm{S}-\mathrm{T}) + j''\mathrm{S} + \alpha\},\ \text{Saturn.}$$

		B		N		R	
$\phi=0$							
j	j''	C	α	C	α	C	α
			°		°		°
1	0	42	179·6	40	180·0	38	180
2		8	0	8	0		
0	1	21	273	51	278·9		
1		13	257	11	255		
0	2	3	297				
2D							
1	0	10	180				
2		5	0				
0	1	4	270				
1		3	257				
0	−1	4	255				
l							
−1	0	6	0	8	0		
1		10	180	8	180		
2		3	180				
0	1	12	263	4	275		
1		3	257				
−1	−1	3	283				
0		12	277	4	265		
$2D-l$							
−2	0	19	180				
1		14	180				
2		4	0				
0	1	6	271				
1		3	257				
−2	−1	5	271				
0		6	267				

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3. In order that the discussion of the various differences may be made clear, it is necessary to state certain facts in the theory of these inequalities which have considerable effect on the final results, and on the methods of obtaining them.

The disturbing function R is of the form—

$$R=\Sigma\{A_0\cos Q+A_1\cos(l+Q)+A_2\cos(2D-l+Q)\\ +A_3\cos(2D+l+Q)+A_4\cos(2D+Q)\\ +A_5\cos(2D-2l+Q)+\ .\ .\ .\ .\ \},$$

where Q here and throughout this discussion is an angle depending only on the solar and planetary arguments, and where the summation sign refers to the different arguments Q.

Let w_1 be the Moon's mean longitude, and c any one of the other five lunar elements. When the value of R has been substituted in the equations of variations and the latter solved we obtain—

$$\delta w_1=\Sigma\{B_0\sin Q+B_1\sin(l+Q)+\ .\ .\ .\ .\ \},$$

$$\delta c=\Sigma\left\{C_0\begin{matrix}\sin\\ \cos\end{matrix}Q+C_1\begin{matrix}\sin\\ \cos\end{matrix}(l+Q)+\ .\ .\ .\ .\ \right\},$$

the term in δc being a sine or cosine according as c is or is not an angular element.

The values of the variations of the elements have to be substituted in—

$$\delta V=\delta w_1+\left(\frac{dV}{dw_1}-1\right)\delta w_1+\frac{dV}{dc}\delta c.$$

Now V consists of a non-periodic term w_1 and periodic terms, say,

$$V=w_1+v_1\sin l+v_2\sin(2D-l)+v_3\sin(2D+l)+\ .\ .\ .\ .\ ,$$

and from this we obtain easily the values of the deviatives of V. The terms which arise from the substitution of the variations of the elements in δV may therefore be divided into two classes. The first of these, which I have called elsewhere* the *primary* terms, consists of those terms in δV which arise from the variation of the non-periodic term; the *secondary* terms are those arising from the substitution in the remaining, that is, the *periodic* terms of V. Thus the values of δw_1 constitute the *primary* terms, and all the other portions of δV the *secondary* terms.

Now the primary terms have the same arguments as the terms in R, that is, the arguments

$$Q,\ l+Q,\ 2D-l+Q,\ .\ .\ .\ .\ ,$$

while the secondary terms have the arguments

$$Q\pm(l),\ Q\pm(2D-l),\ Q\pm(2D+l),\ .\ .\ .\ .$$
$$l+Q\pm(l),\ l+Q\pm(2D-l),\ l+Q\pm(2D+l),\ .\ .\ .\ .\ ,$$

. . . .

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* *The Inequalities in the motion of the moon due to the direct action of the planets*, Pitt Press, Cambridge, 1907.

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the arguments arising from the derivatives of V being enclosed in brackets. It is therefore obvious that the arguments $Q, l+Q$, ... which are present in the primary terms will also appear amongst the secondary terms; they are due to the separation of a product of a sine and a cosine into the sum or difference of two sines.

4. The first result to be noted is the fact that *the sum of the secondaries with arguments* $l+Q-(l), 2D-l+Q-(2D-l)$, ... that is, those having the argument Q, *is very small compared with the coefficient of the primary with argument* Q.* In many cases these secondaries are of the same order of magnitude as the primary of the same argument, the chief of them in general being those arising from the two arguments just written down. This theorem is a consequence of the method of the variation of arbitrary constants. If we had adopted the straightforward method of integrating the original equations for the Moon's motion with the additional value of R due to planetary action, the terms independent of l in the coordinates arising from the terms in R which contain l, and which therefore have the factor e (the lunar eccentricity), would have possessed the factor e^2, while the principal terms due to $R=A_0 \cos Q$ would not have this factor; the former terms must therefore be quite small.

This fact constitutes a useful test of the general accuracy of a large portion of the work, and it was satisfied in all cases as closely as could be expected. It is illustrated in the following table for a few terms of the indirect action of Venus in longitude. The first column gives the arguments of the primaries (that is, of the terms in R), and in line with them are the resulting secondaries for the values of ϕ which stand at the head of each column. The last line but one gives the sums of these secondaries, and the last line the coefficients of the primaries. The sums of these two lines constitute the complete coefficients for the indirect action. The constant angle to be added to each argument is omitted, as it is (in these and in most cases) very nearly the same for each of the numbers in a given column.

$\delta V = 0''\cdot001 C \sin(\phi+\alpha), \phi = iT - i'V$, Venus. Values of C.

Arg. of primary.	T-V	T-2V	2T-2V	3T-2V	3T-3V	4T-3V	5T-3V
$\phi+l$	− 47	− 7	− 92	+ 18	− 13	+ 21	+ 5
$\phi-l$	− 8	...	+ 21	+ 7	+ 4	− 3	+ 7
$\phi+2D-l$	+ 42	+ 6	+ 70	− 20	+ 9	− 18	− 5
$\phi-2D+l$	+ 18	...	+ 3	− 8	...	− 3	− 7
$\phi+2l-2D$	+ 1	...	...	...	+ 1	...	...
$\phi-2l+2D$	+ 2	...	+ 2				
Sums	+ 8	− 1	+ 4	− 3	+ 1	− 3	0
ϕ	− 352	− 14	− 507	+ 275	− 51	+ 155	+ 129

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* This result was not noticed until the computations had been practically completed, when the approximate vanishing of the sums of the secondaries in all cases pointed to a general theorem.

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5. At first sight it would appear that the arguments of the two terms formed, for example, from sums of products like $\sin(l+Q)\cos(2D-l)$, that is, $l+Q\pm(2D-l)$, would have coefficients in δV of the same order of magnitude. It is not so whenever the arguments in R and V contain l, unless the argument in R is of very long period (greater than a score or so of years). This fact might have been predicted from considerations of the same nature as those in the previous paragraph. For example, an argument $2l+Q+(l) = Q+3l$ would not be expected from a direct method of treatment to have a coefficient nearly so large as that of $2l+Q-(l) = Q+l$. The latter is frequently sensible, while there are no terms with the former argument.

This arises, again, from the peculiarities of the method of the variation of arbitrary constants. In the volume referred to above I have shown directly from the method that *if a term of* R *contains* il *in its argument* ($i=1, 2, 3, \ldots$) *and* V *a term* $i'l$($i'=1, 2, 3, \ldots$), *then the secondary arising from the sum of these arguments is always very small compared with that arising from the difference, unless the period of the primary is very long.* This result was proved by inserting the terms due to δl, δe, as given by the equations of variations, in δV. It enabled me to abbreviate the computations by some twenty or more per cent., since it rendered the separate computation of δe unnecessary except in a few easily recognised cases.

6. One other fact should be noted. Although the coefficient of the evection (arg. $2D-l$) is of order $m(1/13)$ compared with that of the principal elliptic term (arg. l), the numerical factors which it contains are so large that in certain functions of the lunar co-ordinates it becomes equal to or of more importance than the principal elliptic term. It is therefore never safe to neglect the former terms unless we know that the latter are quite insensible. This shows most strongly in the arguments $l+\phi$ in R for the indirect action. The terms which multiply $\delta V'$ (the portion of the Earth's longitude due to the action of the planets) are chiefly those due to the evection, while those which multiply $\delta r'$ (the planetary portion of the Earth's radius vector) are due to the evection and the principal elliptic term. Yet the former portions in several cases are larger than the latter.

7. *The differences* B − R. Each of Radau's coefficients depending on the action of Venus is to be diminished by 1/51 of its amount on account of the difference between the value for the mass of Venus adopted by him and that adopted by Newcomb and myself. Each coefficient due to Mercury is to be diminished by 7/60 of its amount for my adopted value of the mass of that planet.

The equations of variations used by Radau appear to require a correction which would increase each coefficient in the ratio 96 : 95, as I have shown in the volume previously referred to. This is confirmed by Newcomb's results, for he obtains values for the principal terms the same as those which I found in the volume quoted.

In the primary terms with arguments independent of the lunar angles, Radau omits the portions due to the term of R which depends on $r^2 \cos 2(V-\mathrm{T})$. These portions constitute about one-eighth of the principal portions for the indirect action, and his coefficients for this portion should therefore be diminished in the ratio 7 : 8. The corresponding correction for the direct action is smaller, and its amount depends on the particular term under consideration.

In the short period terms of the disturbing function for the indirect action of the form $l+\phi$ he omits the portions multiplied by $\delta V'$. These, as stated in No. 6 above, constitute a considerable fraction of the whole. The terms *proportionately* most affected are those with this argument, but the *differences* for the terms of argument $l+\phi-(l)$ are much larger, for the latter (secondary) terms are about three times the former (primary) terms.

The discovery of the causes of divergence between the results of Radau and myself was rendered difficult owing to the fact that, although his secondary inequalities for the Venus terms of argument ϕ are given separately, and are in several cases quite large when the theorem of No. 4 above shows that they ought to be quite small, yet his final coefficients very nearly agreed with mine. As a matter of fact, I have found, from an examination of the separate portions of my results, that these corrections very nearly balance one another in the case of Venus, though they do not do so with the other planets, nor do they in the case of Venus with arguments other than ϕ.

I have compared most of Radau's individual results with mine, and find a close agreement for nearly all those portions which he has computed. For the direct action this agreement was particularly useful, since my coefficients are obtained from formulæ and methods radically different from his, though reducible to them by algebraical processes.

Two other differences should be noticed, namely, those in the terms with arguments $-\mathrm{D}+4\mathrm{T}-3\mathrm{V}$ and $l-\mathrm{D}+4\mathrm{T}-3\mathrm{V}$. These arise from the fact that the indirect action is a little greater and of the opposite sign to the direct action; Radau only takes the latter into account.

8. *The differences* B – N. These have been marked with the letters E_1, E_2, E_3 (explained), and U (unexplained), whenever they exceed $0''{\cdot}020$.

Those marked E_1 are due to the fact that Newcomb substitutes the variations of the elements in the value of V limited to the three terms

$$V=w_1+2e\sin l+\tfrac{5}{4}e^2\sin 2l,$$

instead of taking the complete value; he adds that "in nearly or quite all cases we may drop terms of the second order in e."

This third term of V in reality only produces a few secondaries with coefficients greater than $0''{\cdot}020$, and these arise in the form $2\mathrm{D}-2l+\phi+(2l)=2\mathrm{D}+\phi$; but there are many greater than

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$0''\cdot003$, the adopted limit below which coefficients were dropped. The latter arise also in the forms $l+\phi-(2l)$ and $2\mathrm{D}-l+\phi+(2l)$.

A much more extended and larger set of differences arises, however, from the neglect of the evection in the expression for V. The majority of the differences E_1 due to this cause will be found in the combinations—

$$\begin{aligned}\phi \pm (2\mathrm{D}-l)&,\\ l+\phi+(2\mathrm{D}-l)&=2\mathrm{D}+\phi,\\ 2\mathrm{D}-l+\phi-(2\mathrm{D}-l)&=\phi,\\ 2l-2\mathrm{D}+\phi+(2\mathrm{D}-l)&=l+\phi,\end{aligned}$$

the last being of importance only when $2l-2\mathrm{D}+\phi$ is of long period of the order of ten years, and the first quite rarely. Hence the final coefficients of terms with arguments ϕ, $2\mathrm{D}+\phi$ will be chiefly affected, and a very few in those with arguments $l+\phi$, from this cause. All those differences which may be nearly or altogether explained by the neglect of this and other terms in V are marked E_1. I assume that the terms not present in Newcomb's list have not been computed.

In order to examine the differences E_2 a closer inspection is necessary. On referring to Newcomb's tables for the variations of the elements (p. 154 of his work), I find that under the argument 2D he has only the terms containing (V − T) and 2(V − T) for Venus. There are, however, several other arguments with coefficients of the same order of magnitude as these, and the same fact appears to a smaller extent with the other planets. All differences explained by this cause have been marked E_2.

The small difference E_3 in the great long period term due to Venus scarcely needs explanation. Newcomb has included the portion (less than $0''\cdot10$) due to the mutual perturbations of Venus and the Earth, and he states that his method indicates a possible error of the order $0''\cdot10$. I exclude the former part, but the maximum error by my method should be less than $0''\cdot05$.

Of the differences U there are two in Mars, eight in Jupiter, and one in Saturn. The most important is that in Jupiter with argument J; I have compared the several portions of any value for the primary with that of Radau, and the results (after the corrections noted in No. 7 above have been made) agree for the indirect action, but I have been unable to obtain his result for the direct action from the algebraical formulæ which he gives. The causes for the remaining differences I have not been able to trace, and must leave the values of those coefficients an open question.

9. In working out the terms produced by planetary action, the valuable memoir of M. Radau has been available for comparison of results at almost every stage, and has materially assisted in the prevention of errors made during the course of a piece of work of great complexity, though not of great difficulty when once the theory had been put into final shape. The work of Professor Newcomb only appeared when I had finished the greater part of

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the computations, and would in any case not have been available for detailed comparison without much labour, owing to the complete difference between his method and mine. He combines the direct and indirect actions at the earliest opportunity. I have kept them separate until each was fully completed. The comparison between his results and mine revealed one error in my work which affected the primaries due to the terms with argument ϕ in R by about 5 per cent., and a few of the small secondaries from these terms about twenty per cent.; an error in the equations of variations had almost no effect. These errors have, of course, been corrected in the results given above.

10. Like Professor Newcomb, I have also made an examination of the inequalities of the second order relative to the planetary masses, and have so far found nothing that could sensibly affect the motion of the Moon. I have found an additional portion to the term with argument $l + 3T - 10V$ of the order of $0''\cdot 2$, but the period of this term is so long that it would scarcely affect the observations within the degree of accuracy at present obtainable. The motion of the node of Venus also affects the term with argument $l + 16T - 18V$ by a quantity of the same order of magnitude. The terms of the second order in the Sun's motion, as given in Newcomb's Tables of the Sun,

$$\begin{aligned}\delta V' = & -0''\cdot 265 \cos(4M_1 - 7T_1 + 3V_1) - 0''\cdot 021 \sin(4M_1 - 7T_1 + 3V_1)\\ & + 3\cdot 76 \cos(3J_1 - 8M_1 + 4T_1) + 5\cdot 18 \sin(3J_1 - 8M_1 + 4T_1)\end{aligned}$$

(where the suffix denotes the mean anomalies of the planets instead of the mean motions), or

$$\begin{aligned}& +0''\cdot 266 \sin(1^\circ\cdot 190t + 31^\circ\cdot 8) \quad \text{(period 300 years)}\\ & +6''\cdot 40 \sin(0^\circ\cdot 202t + 231^\circ\cdot 2) \quad \text{(period 1780 years)},\end{aligned}$$

will produce inequalities with the same arguments due to indirect action having the approximate coefficients

$$-0''\cdot 04 \quad \text{and} \quad -0''\cdot 9$$

respectively. These results are sufficiently accurate for tabular purposes, but I shall give a more complete computation, with an examination of the whole effect of the terms of the second order.

New Haven, Conn. U.S.A.:
1907 *November* 26.

Postscript.—Since this paper was sent in, I have computed the terms due to the motion of the ecliptic and have discovered a few new inequalities containing the arguments of the planets. The most important is one with a coefficient $0''\cdot 21$ due to Jupiter, and having a period of 280 years.

1908 *January* 10.